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ON CERTAIN ADDITIONS TO THE THEORY OF ONE SINGULAR INTEGRAL EQUATION*

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Certain additions are given to the theory of a singular integral equation /1/ encountered in some problems of potential theory and in a correspondingly complicated form in two-dimensional elasticity theory.

1. The singular integral equation

$$\frac{1}{\pi i}\int_\gamma \omega(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt=f(t_0),\quad -\rho\leqslant t_0\leqslant\rho,\qquad (1.1)$$
$$-1<\lambda<1$$

is considered, where the free term $f(t)$ is any arbitrarily given Hölder function on a closed real segment $\gamma\,(-\rho\leqslant t\leqslant\rho)$. One of its solutions is

$$\omega(t_0)=\frac{1}{2\pi i}\int_\gamma f(t)\,\chi(\alpha;t,t_0)\left[\frac{1}{t-t_0}-\frac{1}{t-\rho^2/t_0}\right]dt \qquad (1.2)$$
$$\chi(\alpha;t,t_0)=\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{-\alpha}+\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{-\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\alpha}$$
$$\alpha=1-\theta/\pi,\quad \lambda=\cos\theta\quad(0<\theta<\pi)$$

However, the quantity $\omega(t_0)$ given by (1.2) will be an authentic solution of (1.1), continuous in the closed interval $-\rho\leqslant t\leqslant\rho$ and taking zero values at its endpoints $t=\pm\rho$ if the free term is subject to the additional integral condition

$$\Lambda[f(t);\alpha]=\frac{1}{2\pi i}\int_\gamma\left[e^{-\pi i\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\alpha}-e^{\pi i\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{-\alpha}\right]\frac{f(t)}{t}\,dt=0 \qquad (1.3)$$

Meanwhile, another solution of the same Eq.(1.1) that possesses integrable singularities at the endpoints $t=\pm\rho$ and is valid for any free term $f(t)$ has the following more complex form:

$$\mu(t_0)=\frac{1}{2\pi i}\int_\gamma f(t)\,\chi\left(\frac{\theta}{\pi};t,t_0\right)\left(\frac{1}{t-t_0}+\frac{1}{t-\rho^2/t_0}\right)dt\mp \qquad (1.4)$$
$$e^{\pm i\theta}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\mp\theta/\pi}\Lambda\left[f(t);\frac{\theta}{\pi}\right]+2i\sin\theta\,\times$$
$$B(\vartheta)\left[e^{-i\theta}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\theta/\pi}+e^{i\theta}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{-\theta/\pi}\right],\quad -\rho<t_0<\rho$$

We note that the binomial in the square brackets with the arbitrary constant factor $B(\theta)$ is a solution of the homogeneous Eq.(1.1) (for $f(t_0)=0$).

If the free term $f(t_0)$ of (1.1) satisfies condition (1.3), then the solutions (1.2) and (1.4) can be distinguished only by a certain solution of the homogeneous Eq.(1.1).

2. We study the fractional-linear power functions in (1.2) and (1.4). We expand the former (in the neighbourhood of the remote part of the domain) in non-positive powers of the affix z. As is clear, an expansion holds for the variable z which exceeds the quantity ρ in absolute value (the asterisk suerpscript on the summation sign in the second sum indicates that adjacent values of the exponent ν differ by two units)

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha}=\sum_{k=0}^{\infty}\frac{(\pm\alpha)^k}{k!}\,r^k(z), \qquad (2.1)$$
$$r(z)=\ln\frac{\rho-z}{-\rho-z}=-2\frac{\rho}{z}\sum_{\nu=0}^{\infty}{}^{*}\,\frac{1}{1+\nu}\left(\frac{\rho}{z}\right)^{\nu}$$

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It reduces to the following:

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha}=p\left(\alpha^2;\frac{\rho}{z}\right)\pm\alpha q\left(\alpha^2;\frac{\rho}{z}\right) \tag{2.2}$$

in which the components are given, in turn, by the expansions $(a_n(\alpha^2)$ are quantities explained later)

$$p\left(\alpha^2;\frac{\rho}{z}\right)=\sum_{\nu=0}^{\infty}a_{2\nu}(\alpha^2)\left(\frac{\rho}{z}\right)^{2\nu},\quad q\left(\alpha^2;\frac{\rho}{z}\right)=\sum_{\nu=0}^{\infty}a_{2\nu+1}(\alpha^2)\left(\frac{\rho}{z}\right)^{2\nu+1} \tag{2.3}$$

On the basis of auxiliary relationships (the coefficients therein are calculated comparatively easily in conformity with (2.1))

$$r^k(z)=(-2)^k\left(\frac{\rho}{z}\right)^k\sum_{\nu=0}^{\infty}{}^{*}\,b_\nu^{(k)}\left(\frac{\rho}{z}\right)^\nu\quad(k=1,2,\ldots),\quad b_\nu^{(1)}=\frac{1}{1+\nu} \tag{2.4}$$

the initial formula (2.1) is converted in an elementary manner into the following double sum (the exponent ν takes values of identical evenness with k):

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha}=\sum_{k=0}^{\infty}(-2)^k\frac{(\pm\alpha)^k}{k!}\sum_{\nu=k}^{\infty}{}^{*}\,b_{\nu-k}^{(k)}\left(\frac{\rho}{z}\right)^\nu$$

Changing the orders of summation therein, we arrive at the expansion

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha}=\sum_{\nu=0}^{\infty}\left(\frac{\rho}{z}\right)^\nu\sum_{k=0,1}^{\nu}{}^{*}\,\frac{(\pm\alpha)^k}{k!}(-2)^k b_{\nu-k}^{(k)}$$

The values of the exponent k (over which the inner summation is performed) are here taken in identical evenness with ν. Separating this expansion in conformity with (2.3), we obtain

$$a_{2\nu}(\alpha^2)=\sum_{k=0}^{\nu}(-2)^{2k}\frac{\alpha^{2k}}{2k!}b_{2(\nu-k)}^{(2k)},\quad a_{2\nu+1}(\alpha^2)=\sum_{k=0}^{\nu}(-2)^{2k+1}\frac{\alpha^{2k}}{(2k+1)!}b_{2(\nu-k)}^{(2k+1)}$$

Then using the notation

$$c_{2\nu}^{\pm}(\alpha^2)=c_{2\nu}(\alpha)=a_{2\nu}(\alpha^2),\quad c_{2\nu+1}^{\pm}(\alpha)=\pm\alpha a_{2\nu+1}(\alpha^2)$$

we give the following form to (2.2)

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha}=\sum_{\nu=0}^{\infty}c_\nu^{\pm}(\alpha)\left(\frac{\rho}{z}\right)^\nu \tag{2.5}$$

Computation of the required coefficients $a_\nu(\alpha^2)$ requires a knowledge of the suitable $b_m^{(n)}$; to find these an elementary recursion formula is easily deduced. To do this, we introduce a function regular in the circle $|z|<1$

$$h(z)=\int_{-R}^{-1}\frac{dt}{t-z}+\int_{1}^{R}\frac{dt}{t-z}\qquad\text{as}\qquad R\to\infty$$

It can be given adequately as

$$h(z)=-\ln\frac{1-z}{1+z}=2z\sum_{\nu=0}^{\infty}{}^{*}\,\frac{z^\nu}{1+\nu} \tag{2.6}$$

Hence, we again find that the coefficients in (2.4) are representable in integral form

$$(-2)^k b_\nu^{(k)}=\frac{1}{2\pi i}\int_\gamma\ln^k\frac{1-z}{1+z}\frac{dz}{z^{k+\nu+1}};\quad k=1,2,\ldots;\quad\nu=0,2,4,.. \tag{2.7}$$

where γ is some closed contour delimiting a certain neighbourhood $z=0$ with a counterclockwise direction of traversal. Taking this integral by parts, we arrive at the formula

$$(-2)^k b_\nu^{(k)} = -\frac{k}{k+\nu}\frac{1}{\pi i}\int\limits_\gamma \ln^{k-1}\frac{1-z}{1+z}\frac{dz}{(1-z^2)\,z^{k+\nu}}$$

Furthermore, comparing (2.17) with the equalities (2.6) and (2.4), we write the last formula in the form

$$b_\nu^{(k)} = -\frac{k}{k+\nu}\frac{1}{2\pi i}\int\limits_\gamma \sum_{n=0}^{\infty}{}^{*} b_n^{(k-1)} z^n \sum_{m=0}^{\infty}{}^{*} z^m \frac{dz}{z^{\nu+1}} =$$
$$-\frac{k}{k+\nu}\frac{1}{2\pi i}\int\limits_\gamma \sum_{m=0}^{\infty}{}^{*} z^m \sum_{n=0}^{m} b_n^{(k-1)}\frac{dz}{z^{\nu+1}}$$

from which we rapidly arrive at the required ultimately simple formula

$$b_\nu^{(k)} = \frac{k}{k+\nu}\sum_{n=0}^{\nu}{}^{*} b_n^{(k-1)};\quad b_0^{(\nu)} = 1,\ \nu \geqslant 0,\quad b_\nu^{(1)} = \frac{1}{\nu+1},\quad \nu > 0 \tag{2.8}$$

From it we successively find all the necessary coefficients $b_\nu^{(k)}$.

3. It is sometimes desirable to have expressions for the coefficients of the expansions of the fractional-linear power functions with the exponent $\pm\vartheta/\pi\ (\pm\alpha)$ in terms of the coefficients of expansions of the same functions with exponent $\pm\alpha\ (\pm\vartheta/\pi)$. We achieve this by starting from the obvious equation

$$I_n\left(\pm\frac{\vartheta}{\pi}\right) = \frac{1}{2\pi i}\int\limits_\gamma \left(\frac{\rho-t}{-\rho-t}\right)^{\pm\vartheta/\pi}\left(\frac{t}{\rho}\right)^n\frac{dt}{t} = \pm\frac{i}{2}\frac{e^{\pm i\vartheta}}{\sin\vartheta}C_n^{\pm}\left(\frac{\vartheta}{\pi}\right) \tag{3.1}$$

or its equivalent (C_R is a cirle of radius R)

$$\pm\frac{i}{2}\frac{e^{\pm i\vartheta}}{\sin\vartheta}\left[\frac{1}{2\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\vartheta/\pi}\left(\frac{t}{\rho}\right)^n\frac{dt}{t} - C_n^{\pm}\left(\frac{\vartheta}{\pi}\right)\right] = 0 \tag{3.2}$$

On the other hand, it is evident that

$$\frac{1}{2\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\vartheta/\pi}\left(\frac{t}{\rho}\right)^n\frac{dt}{t} = \frac{1}{2\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm1}\left(\frac{t}{\rho}\right)^n\frac{dt}{t} \tag{3.3}$$

By barely modifying the writing of the second factor in the last integral, we obtain

$$\frac{1}{2\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\left(1\mp\frac{2\rho}{t\pm\rho}\right)\left(\frac{t}{\rho}\right)^n\frac{dt}{t} = C_n^{\mp}(\alpha) + \Lambda_n(\mp\alpha) \tag{3.4}$$

$$\Lambda_n(\mp\alpha) = \mp\frac{1}{\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\frac{\rho}{t\pm\rho}\left(\frac{t}{\rho}\right)^n\frac{dt}{t} = \tag{3.5}$$
$$\mp\frac{1}{\pi i}\int\limits_{C_{R\to\infty}}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\sum_{k=0}^{\infty}(\mp1)^k\left(\frac{\rho}{t}\right)^{k+1}\left(\frac{t}{\rho}\right)^n\frac{dt}{t}$$

By firtue of the fundamental formula (2.5), we arrive at the following expansion of the integrand in the last integral

$$\sum_{k=0}^{\infty}(\mp1)^k\left(\frac{\rho}{t}\right)^{k+1}\sum_{\nu=0}^{\infty}C_\nu^{\mp}(\alpha)\left(\frac{\rho}{t}\right)^\nu = \sum_{\nu=1}^{\infty}\left(\frac{\rho}{t}\right)^\nu\sum_{k=0}^{\nu-1}(\mp1)^k C_{\nu-k-1}^{\mp}(\alpha)$$

Now, we again find from (3.5)

$$\Lambda_n(\mp\alpha) = \mp2\sum_{k=0}^{n-1}(\mp1)^k C_{n-k-1}^{\mp}(\alpha) \tag{3.6}$$

We have for the integral on the left-hand side of (3.3)

$$C_n^{\mp}(\alpha) \mp 2\sum_{k=0}^{n-1}(\mp1)^k C_{n-k-1}^{\mp}(\alpha)$$

Hence, taking (3.2) into account, we obtain after replacing $k+1$ by the exponent ν

$$C_n^{\pm}\left(\frac{\vartheta}{\pi}\right) = \mp 2\sum_{\nu=1}^{n}(\mp 1)^{\nu-1} C_{n-\nu}^{\mp}(\alpha) + C_n^{\mp}(\alpha) = \tag{3.7}$$

$$2\sum_{\nu=0}^{n}(\mp 1)^{\nu} C_{n-\nu}^{\mp}(\alpha) - C_n^{\mp}(\alpha)$$

We later consider separately the case of even and odd n. In the former, when ν is even $(\mp 1)^{\nu} C_{n-\nu}^{\mp}(\alpha) = C_{n-\nu}^{+}(\alpha) = C_{n-\nu}(\alpha)$; if ν is odd $(\mp 1)^{\nu} C_{n-\nu}^{\mp}(\alpha) = C_{n-\nu}^{+}(\alpha) = C_{n-\nu}(\alpha)$ and (3.7) takes the form

$$C_n\left(\frac{\vartheta}{\pi}\right) = 2\sum_{\nu=0}^{n} C_{n-\nu}(\alpha) - C_n(\alpha)$$

Now let n be an odd number. Then for even ν

$$(\mp 1)^{\nu} C_{n-\nu}^{\mp}(\alpha) = C_{n-\nu}^{\mp}(\alpha) = \mp C_{n-\nu}(\alpha)$$

and for odd ν

$$(\mp 1)^{\nu} C_{n-\nu}^{\mp}(\alpha) = \mp C_{n-\nu}^{\mp}(\alpha) = \mp C_{n-\nu}(\alpha),\ C_n^{\pm}(\vartheta/\pi) = \pm C_n(\vartheta/\pi)$$

and relationship (3.7) is written thus

$$-C_n\left(\frac{\vartheta}{\pi}\right) = 2\sum_{\nu=0}^{n} C_{n-\nu}(\alpha) - C_n(\alpha) \tag{3.8}$$

4. If the functions $f(t)$ in (1.1) are not intrinsic elements of indeterminacy (yielding to at least partial elimination in connection with the requirement to conserve condition (1.4)), then (1.1) will not generally possess a solution that is continuous to the endpoints $t = \pm\rho$. In this case, it is best to select the step-by-step means of its examination that would, in the long run, enable us to arrive at a form of the solution with explicitly separated continuous and unbounded components. We first write the continuous solution $\omega(t)$ of (1.1) with a modified right-hand side equal to $f(t) - N$, where N is a constant fixed in conformity with (1.4). Then we take the solution of this same equation that contains the already known constant N on the right. Proceeding as in /2/ (with modifications introduced somewhere, it is difficult to foresee them in advance), we obtain

$$\mu_0(t_0) = iN\left\{\frac{1}{2\sin\theta}\left[e^{i\theta}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{-\theta/\pi} - e^{-i\theta}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\theta/\pi}\right] \pm \right. \tag{4.1}$$

$$\left. e^{\pm i\theta}\operatorname{ctg}\theta\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\mp\theta/\pi}\right\}$$

Meanwhile, any of the expressions

$$\mu_0(t_0) = \mp iNe^{\mp i\theta}\operatorname{ctg}\frac{\theta}{2}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\pm\theta/\pi} \tag{4.2}$$

can also be taken as such a solution.

We can arrive at it by removing the appropriate solution of the homogeneous Eq.(1.1) from (4.1) (as is seen, both solutions, which are given by the last equalities, differ by a certain solution of the same homogeneous Eq.(1.1)).

We note that the validity of (4.1) and (4.2) can be verified by relying on the relationship

$$\frac{1}{\pi i}\int_{\gamma}\left(\frac{\rho - t}{-\rho - t}\right)^{\pm\theta/\pi}\left(\frac{1}{t - t_0} - \lambda\frac{1}{t - \rho^2/t_0}\right)dt = \pm ie^{\pm i\theta}\operatorname{tg}\frac{\theta}{2} \tag{4.3}$$

Appending a solution of the homogeneous Eq.(1.1) still to the sum of $\omega_0(t)$ and $\mu_0(t)$, we compile the general solution of the inhomogeneous Eq.(1.1) in the desired split mode.

5. The reasoning in the execution of the overwhelming number of operations referring to setting up (4.1) and (4.3) is carried over to the case of suitable curvilinear segments with the same endpoints $t = \pm\rho$ above or below the rectilinear segment $(-\rho, \rho)$ with just insignificant changes. Doubts can occur only in connection with the calculations associated with deriving the reducible formula

$$\int_{\substack{\gamma \\ z \to t_0}} \frac{dt}{t - z} \mp \pi i = \int_{\substack{\gamma \\ z \to t_0}} \frac{dt}{t - \rho^2/z} \tag{5.1}$$

where the second component on the left is taken with the minus or plus sign, depending on whether the variable z tends upward or downward to the point t_0 of the segment γ.

This formula was derived under the assumption that γ is a segment of the real axis symmetric about the origin. In connection with the above, the question of interest is whether (5.1) remains valid for certain (and precisely which) curvilinear segments with the endpoints $t=\pm\rho$.

Let γ^* later be a curvilinear segment connecting the points $t=\pm\rho$ as before and located in the upper half-plane with the same direction of traversal as γ (from the left endpoint $t=-\rho$ to the right $t=\rho$). The relationship (5.1) is analytically continuable in a domain bounded by the segments γ and γ^* and (on the basis of Cauchy's theorem), we express the integrals with respect to γ on both its sides in terms of integrals with respect to the other curvilinear segments γ^*. Then letting $z\to t_0$, where t_0 is the affix of the segment γ^*, we arrive at the relationship

$$\int_{\gamma^*}\frac{dt}{t-t_0}=\int_{\gamma^*}\frac{dt}{t-\rho^2/t_0} \tag{5.2}$$

By virtue of Cauchy's theorem, we will have

$$\int_{\gamma^*}\frac{dt}{t-z}-\pi i=\int_{\gamma^*}\frac{dt}{t-\rho^2/z}$$

for the variable z located above the curve γ^* (which assumes the continuation of (5.1) into the domain enclosing the segment γ^*).

We then pass to the limit $z\to t_0$ where the point t_0 is again taken on γ^*, we invariably return to the same relationship (5.2).

We arrive at the identical result by considering the arc γ^* located in the lower half-plane.

The assumption made provides the possibility of directly arriving at a treatment of a singular equation with an arbitrary contour of integration, but relying on the method proposed below.

6. We briefly consider the singular integral equation

$$\frac{1}{\pi i}\int_L\mu(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt=f(t_0) \tag{6.1}$$

in which integration is over a fairly smooth curvilinear segment L connecting the points $t=\pm\rho$ and located above or below the segment γ (for the traversal direction from the endpoint $t=-\rho$ to $t=\rho$).

Its inversion can be approached differently. Indeed, we first write (6.1) in the form

$$\mu(t_0)+\frac{1}{\pi i}\int_{\substack{L\\ z=t_0}}\mu(t)\left[\frac{1}{t-z}-\lambda\frac{1}{t-\rho^2/z}\right]dt=f(t_0)$$

where z is in a domain bounded by the curve L and the diameter $(-\rho,\rho)$ and then again in the form

$$\delta(t_0)+\frac{1}{\pi i}\int_{\substack{L\\ z\to t_0}}\delta(t)\left[\frac{1}{t-z}-\lambda\frac{1}{t-\rho^2/z}\right]dt=F(t_0)$$

$$F(z)=-\frac{1}{\pi i}\int_L f(t)\left[\frac{1}{t-z}-\lambda\frac{1}{t-\rho^2/z}\right]dt,\quad \delta(t)=\mu(t)-f(t)$$

The penultimate equality is analytically continuable in the domain mentioned and we then again pass to the limit $z\to t_0$ in it, where t_0 is already a point of the diameter $(-\rho,\rho)$. We will have

$$\delta(t_0)+\frac{1}{\pi i}\int_L\delta(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt=F(t_0)$$

Finally, taking into account the formulas

$$\frac{1}{\pi i}\int_L\frac{\delta(t)}{t-t_0}dt=-\delta(t_0)+\frac{1}{\pi i}\int_{-\rho}^{\rho}\frac{\delta(t)}{t-t_0}dt \qquad \frac{1}{\pi i}\int_L\frac{\delta(t)}{t-\rho^2/t_0}dt=\frac{1}{\pi i}\int_{-\rho}^{\rho}\frac{\delta(t)}{t-\rho^2/t_0}dt$$

which are valid on the same diameter, we arrive at the singular equation studied

$$\frac{1}{\pi i}\int_{-\rho}^{\rho}\delta(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt=F(t_0),\quad -\rho<t<\rho \tag{6.2}$$

Borrowing the expression for $\delta(t)$ from (6.2) (with or without an appropriate solvability condition) and again returning (by means of continuation) to the initial contour L, we find the required density $\mu(t)$.

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STABILITY OF A GROWING VISCOELASTIC ROD SUBJECTED TO AGEING*

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The stability of a compressed growing rod of viscoelastic material that possesses the property of ageing /1/ is investigated. In conformity with the Chatayev definition of the stability of dynamic systems and the Lyapunov method described in /2/, stability conditions are obtained for a rod growing during a finite time interval, and in finite and semi-infinite time intervals. Some results of a numerical analysis of the behaviour of such a rod are presented in /3/.

1. Variational formulation of the problem of the stability of a growing viscoelastic rod. We consider a rod that grows in both the longitudinal and transverse directions, where its transverse section possesses two axes of symmetry at each time. The law of variation in the rod length as well as the kinematics of its growth in the plane of the transverse section are considered given /1/, whereupon the time of material generation $\tau^*(\rho)$ can be determined in the neighbourhood of a point with coordinates $\rho=\{x,y,z\}$ (the rod length is $l(0)$ at the initial instant). As the time t_1 elapses, the length, the cross-sectional area, and the moment of inertia of the rod remain unchanged and respectively equal to l_0, $F_0(x)$, $J_0(x)$ $(F_0(x)\neq 0,\ J_0(x)\neq 0)$. A one-parameter conservative compressive load $q(t,x)$ acts on the rod, and causes the normal force $N_0(t,x)=-\beta N_*(t,x)$ therein (β is the load parameter). Axial displacements $u_0(t,x)$, determining the trajectory of the unperturbed motion, appear in the rod subjected to the load in the rectilinear equilibrium position. We assume that when there is no external load the rod axis has a small initial curvature $\alpha w_0(x)$ in the xy plane (α is a small parameter). In this case the rod receives additional displacement αu_1, αw_1 under the effect of the load. We designate the rod motion to which the displacements $u_0+\alpha u_1$, αw_1 correspond to perturbed, and αu_1, αw_1 as perturbations. The rod curvature αw_0 is an external perturbation, with respect to which it is assumed that it is twice differentiable with respect to x, where the first and the second derivatives are square summable in the segment $[0,l(t)]$, where $l(t)$ is the rod length at the running time t.

Definition. The unperturbed rod motion is called stable with respect to the perturbation

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